

ON THE CONTRACTION OF $so(4)$ TO $iso(3)$

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ABSTRACT. For any skew-Hermitian integrable irreducible infinite dimensional representation η of $iso(3)$, we find a sequence of (finite dimensional) irreducible representations ρ_n of $so(4)$ which contract to η .

1. INTRODUCTION

One of the first known examples of contraction of Lie algebra representations, given in the early work of İnönü and Wigner [1], is the contraction of the representations of the Lie algebra $so(3)$ to those of $iso(2)$. In that example, starting from a sequence, $\{\rho_j\}_{j=1}^{\infty}$ of finite dimensional representations of $so(3)$ with increasing dimension, they obtained an infinite dimensional representation, η_q of $iso(2)$. They proved the contraction of the representations by the following type of convergence of matrix elements:

$$\lim_{j \rightarrow \infty} \langle {}^j_{m'} | \rho_j(X) | {}^j_m \rangle = \langle {}^q_{m'} | \eta_q(X) | {}^q_m \rangle \quad (1.1)$$

where $|{}^j_m\rangle$ (respectively $|{}^q_m\rangle$) is an element in an orthonormal basis of ρ_j (respectively η_q).

In this paper we show that the same type of convergence of matrix elements as in (1.1), holds for the contraction of the finite dimensional irreducible representations of $so(4)$ to infinite dimensional irreducible representations of $iso(3)$. The convergence is proved using a less familiar description of the irreducible representations of $so(4)$ and $iso(3)$, due to Pauli [2].

Our paper is divided as follows: In sections 2 and 3 we will describe the representation theory of $so(4)$ and $iso(3)$ respectively. In section 4 we give the contraction of the algebra $so(4)$ to $iso(3)$ and prove the convergence of the appropriate matrix elements.

2. REPRESENTATION THEORY OF $so(4)$.

In this section and the one that follows we describe all the skew-Hermitian irreducible finite dimensional representations of $so(4)$ and all the skew-Hermitian irreducible infinite dimensional representations of $iso(3)$. We recall that the Lie algebra $so(4)$ is the direct sum of two copies of the Lie algebra $so(3)$.

Moreover every irreducible representation of $so(4)$ is a tensor product of two irreducible representations of $so(3)$. From the work of Weimar-Woods [3, 4] we know all the contractions of representations of $so(3)$ and hence we also know all the contractions of representations of $so(4)$ that respect the decomposition $so(4) = so(3) \oplus so(3)$. As noted in [3], the contraction of $so(4)$ to $iso(3)$ does not respect this decomposition. Hence we use another description of the representations of $so(4)$ which was given by Pauli [2]. The resemblance of the representations of $so(4)$ and $iso(3)$ in this description is more convenient for the contraction procedure. We also give the relation between the parameterization of the irreducible representations as was given by Pauli [2] and the more usual parameterization as a tensor product of two irreducible representations of $so(3)$.

The Lie algebra $so(4)$ can be defined by the basis $\{M_1, M_2, M_3, N_1, N_2, N_3\}$ satisfying the following commutation relations:

$$[M_i, M_j] = i\epsilon_{ijk}M_k \quad (2.1)$$

$$[N_i, N_j] = i\epsilon_{ijk}M_k \quad (2.2)$$

$$[M_i, N_j] = i\epsilon_{ijk}N_k \quad (2.3)$$

where ϵ_{ijk} is the Levi-Civita totally antisymmetric symbol. We will describe all the irreducible finite dimensional integrable representations of $so(4)$ in terms of another basis which is $\{M_+, M_-, M_3, N_+, N_-, N_3\}$, where $M_\pm = M_1 \pm iM_2$, $N_\pm = N_1 \pm iN_2$. $so(4)$ has two independent invariants (Casimir operators) $\vec{M} \cdot \vec{N}$ and $\frac{1}{2}(M^2 + N^2)$ ¹. On each irreducible representation of $so(4)$, $\vec{M} \cdot \vec{N}$ and $M^2 + N^2$ act as scalar operators with scalars which we denote by G and F respectively. These two scalars determine uniquely (up to an isomorphism) the irreducible representation of $so(4)$. We denote the irreducible representation of $so(4)$ with F and G by $\rho_{F,G} : so(4) \rightarrow gl(V_{F,G})$. The representation space $V_{F,G}$ has an orthonormal basis of the form $\left\{ |_{j,m_j}^{F,G} \rangle : j \in \{j_0, j_0 + 1, \dots, n\}, m_j \in \{-j, -j + 1, \dots, j\} \right\}$, where $j_0, n \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ and they satisfy $G^2 = j_0^2(n+1)^2$, $2F = j_0^2 + (n+1)^2 - 1$. The dimension of $V_{F,G}$ is given by $\sum_{i=j_0}^n (2i+1) = (n+1)^2 - j_0^2$. The representation $\rho_{F,G}$ is given by:

$$\rho_{F,G}(M_3)|_{j,m_j}^{F,G} \rangle = m_j |_{j,m_j}^{F,G} \rangle \quad (2.4)$$

$$\rho_{F,G}(M_\pm)|_{j,m_j}^{F,G} \rangle = \sqrt{(j \mp m_j)(j \pm m_j + 1)} |_{j,m_j \pm 1}^{F,G} \rangle \quad (2.5)$$

¹ $\vec{M} \cdot \vec{N}$ and $M^2 + N^2$ are elements of the center of the universal enveloping algebra of $so(4)$ and are given by: $\vec{M} = (M_1, M_2, M_3)$, $\vec{N} = (N_1, N_2, N_3)$, $\vec{M} \cdot \vec{N} = \sum_{i=1}^3 M_i N_i$, $M^2 + N^2 = \sum_{i=1}^3 (M_i^2 + N_i^2)$

$$\begin{aligned} \rho_{F,G}(N_3)|_{j,m_j}^{F,G}\rangle = & \alpha_j^{F,G} \sqrt{(j+m_j)(j-m_j)}|_{j-1,m_j}^{F,G}\rangle \\ & + \beta_j^{F,G} m_j |_{j,m_j}^{F,G}\rangle + \\ & \alpha_{j+1}^{F,G} \sqrt{(j+m_j+1)(j-m_j+1)}|_{j+1,m_j}^{F,G}\rangle \end{aligned} \quad (2.6)$$

$$\begin{aligned} \rho_{F,G}(N_{\pm})|_{j,m_j}^{F,G}\rangle = & \pm \alpha_j^{F,G} \sqrt{(j \mp m_j)(j \mp m_j - 1)}|_{j-1,m_j \pm 1}^{F,G}\rangle \\ & + \beta_j^{F,G} \sqrt{(j \mp m_j)(j \pm m_j + 1)}|_{j,m_j \pm 1}^{F,G}\rangle \\ & \mp \alpha_{j+1}^{F,G} \sqrt{(j \pm m_j + 1)(j \pm m_j + 2)}|_{j+1,m_j \pm 1}^{F,G}\rangle \end{aligned} \quad (2.7)$$

where

$$\beta_j^{F,G} = \frac{G}{j(j+1)} \quad (2.8)$$

$$\alpha_j^{F,G} = \sqrt{\frac{2F+1-j^2-\frac{G^2}{j^2}}{(2j+1)(2j-1)}} \quad (2.9)$$

2.1. $so(4)$ as the direct sum $so(3) \oplus so(3)$.

We define $K_i \equiv \frac{1}{2}(M_i + N_i)$, $L_i \equiv \frac{1}{2}(M_i - N_i)$, $i = 1, 2, 3$ and we get a new basis for $so(4)$, $\{K_1, K_2, K_3, L_1, L_2, L_3\}$, satisfying the following commutation relations:

$$[K_i, K_j] = i\epsilon_{ijk}K_k \quad (2.10)$$

$$[L_i, L_j] = i\epsilon_{ijk}L_k \quad (2.11)$$

$$[K_i, L_j] = 0. \quad (2.12)$$

We see that either $\{K_1, K_2, K_3\}$ or $\{L_1, L_2, L_3\}$ span an ideal of $so(4)$, which is isomorphic to $so(3)$ and hence, $so(4) = so(3) \oplus so(3)$. The invariant operators in terms of this basis are:

$$\vec{M} \cdot \vec{N} = K^2 - L^2 \quad (2.13)$$

$$\frac{1}{2}(M^2 + N^2) = K^2 + L^2 \quad (2.14)$$

It is well known² that each irreducible finite dimensional representation of $so(4)$ is a tensor product of two irreducible finite dimensional representations of $so(3)$. So for each irreducible representation $\rho_{F,G} : so(4) \rightarrow gl(V_{F,G})$ there are some $k, l \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ such that the representation $\rho^k \otimes \rho^l : so(3) \oplus so(3) \rightarrow gl(V_k \otimes V_l)$ is isomorphic to $\rho_{F,G}$. The representation

²For example [5].

(ρ^j, V_j) is the unique³ irreducible representation of $so(3)$ with dimension $2j + 1$.

For the irreducible representation of $so(4)$ from Pauli's description, $\rho_{F,G}$, which is isomorphic to $\rho^k \otimes \rho^l$, we have the following relations:

$$G = k(k+1) - l(l+1) = \pm j_0(n+1) \quad (2.15)$$

$$F = k(k+1) + l(l+1) = \frac{j_0^2 + (n+1)^2 - 1}{2} \quad (2.16)$$

$$\dim V_{F,G} = \dim(V_K \otimes V_l) = (2k+1)(2l+1) = \quad (2.17)$$

$$(n+1)^2 - j_0^2 = \left(2\sqrt{1+2(F+G)} - 1\right) \left(2\sqrt{1+2(F-G)} - 1\right)$$

$$n = k + l = -2 + \sqrt{1+2(F+G)} + \sqrt{1+2(F-G)} \quad (2.18)$$

$$j_0 = |k - l| = |\sqrt{1+2(F+G)} - \sqrt{1+2(F-G)}| \quad (2.19)$$

$$k = -1 + \sqrt{1+2(F+G)} \quad (2.20)$$

$$l = -1 + \sqrt{1+2(F-G)} \quad (2.21)$$

$$G > 0 \implies l < k, k = \frac{n+j_0}{2}, l = \frac{n-j_0}{2} \quad (2.22)$$

$$G < 0 \implies l > k, k = \frac{n-j_0}{2}, l = \frac{n+j_0}{2} \quad (2.23)$$

$$G = 0 \implies l = k = \frac{n}{2}, j_0 = 0 \quad (2.24)$$

The two pairs of parameters (k, l) and (F, G) are equivalent and knowing the value of one of these pairs determines uniquely the irreducible representation. The pair (j_0, n) does not determine uniquely the irreducible representation, but the values of (j_0, n) along with the knowledge of the sign of G does.

3. REPRESENTATION THEORY OF $iso(3)$

The Lie algebra $iso(3)$ can be defined by the basis $\{J_1, J_2, J_3, P_1, P_2, P_3\}$ satisfying the following commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad (3.1)$$

$$[P_i, P_j] = 0 \quad (3.2)$$

$$[J_i, P_j] = i\epsilon_{ijk}P_k \quad (3.3)$$

We will describe all the skew-hermitian irreducible integrable infinite dimensional representations of $iso(3)$ in the basis $\{J_+, J_-, J_3, P_+, P_-, P_3\}$ where $J_{\pm} = J_1 \pm iJ_2$, $P_{\pm} = P_1 \pm iP_2$. $iso(3)$ has two independent invariants

³There is only one for each positive integer dimension, up to an isomorphism of representations and these are all the finite dimensional irreducible representations of $so(3)$. See for example [6].

(Casimir operators) P^2 and $\vec{J} \cdot \vec{P}$. On each irreducible representation of $iso(3)$, P^2 and $\vec{J} \cdot \vec{P}$ act as scalar operators with the scalars which we denote by p^2 and C respectively. These two scalars determine uniquely (up to an isomorphism) the irreducible representation of $iso(3)$. We denote the irreducible representation of $iso(3)$ with given p^2 and C by $\eta_{p^2, C} : iso(3) \rightarrow gl(W_{p^2, C})$. The representation space $W_{p^2, C}$ has an orthonormal basis of the form $\{|_{j, m_j}^{p^2, C}\rangle : j \in \{j_0, j_0 + 1, \dots\}, m_j \in \{-j, -j + 1, \dots, j\}\}$, where $j_0 \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ and they satisfy $C^2 = j_0^2 p^2$. All the $W_{p^2, C}$ are infinite dimensional. The representation $\eta_{p^2, C}$ is given by:

$$\eta_{p^2, C}(J_3)|_{j, m_j}^{p^2, C}\rangle = m_j|_{j, m_j}^{p^2, C}\rangle \quad (3.4)$$

$$\eta_{p^2, C}(J_{\pm})|_{j, m_j}^{p^2, C}\rangle = \sqrt{(j \mp m_j)(j \pm m_j + 1)}|_{j, m_j \pm 1}^{p^2, C}\rangle \quad (3.5)$$

$$\begin{aligned} \eta_{p^2, C}(P_3)|_{j, m_j}^{p^2, C}\rangle = & \quad \tilde{\alpha}_j^{p^2, C} \sqrt{(j + m_j)(j - m_j)}|_{j-1, m_j}^{p^2, C}\rangle \\ & + \tilde{\beta}_j^{p^2, C} m_j |_{j, m_j}^{p^2, C}\rangle + \\ & \tilde{\alpha}_{j+1}^{p^2, C} \sqrt{(j + m_j + 1)(j - m_j + 1)}|_{j+1, m_j}^{p^2, C}\rangle \end{aligned} \quad (3.6)$$

$$\begin{aligned} \eta_{p^2, C}(P_{\pm})|_{j, m_j}^{p^2, C}\rangle = & \quad \pm \tilde{\alpha}_j^{p^2, C} \sqrt{(j \mp m_j)(j \mp m_j - 1)}|_{j-1, m_j \pm 1}^{p^2, C}\rangle \\ & + \tilde{\beta}_j^{p^2, C} \sqrt{(j \mp m_j)(j \pm m_j + 1)}|_{j, m_j \pm 1}^{p^2, C}\rangle \\ & \mp \tilde{\alpha}_{j+1}^{p^2, C} \sqrt{(j \pm m_j + 1)(j \pm m_j + 2)}|_{j+1, m_j \pm 1}^{p^2, C}\rangle \end{aligned} \quad (3.7)$$

where

$$\tilde{\beta}_j^{p^2, C} = \frac{C}{j(j+1)} \quad (3.8)$$

$$\tilde{\alpha}_j^{p^2, C} = \sqrt{\frac{p^2 - \frac{C^2}{j^2}}{(2j+1)(2j-1)}} \quad (3.9)$$

4. CONTRACTION OF THE MATRIX ELEMENTS

In this section, we first recall the definition for contraction and give the contraction of the algebra $so(4)$ to $iso(3)$. Then, for each of the representations $\eta_{p^2, C}$ we specify a suitable sequence of the representations $\rho_{F(n), G(n)}$ such that we obtain the desired convergence of matrix elements. We will not address the question of contraction of the group representations which was solved by Dooley and Rice [7] and was considered by others [8, 9, 10]. We note that a contraction of the representations of $so(3, 1)$ to those of $iso(3)$

was done by Weimar-Woods [11].

4.1. Contraction of $so(4)$ to $iso(3)$.

We recall the formal definition for a contraction of Lie algebras. Our notations are similar to those of Weimar-Woods [12].

Definition 1. Let U be a complex or real vector space. Let $\mathcal{G} = (U, [_, _])$ be a Lie algebra with Lie product $[_, _]$. For any $\epsilon \in (0, 1]$ let $t_\epsilon \in \text{Aut}(U)$ (t_ϵ is a linear invertible operator on U) and for every $X, Y \in U$ we define

$$[X, Y]_\epsilon = t_\epsilon^{-1}([t_\epsilon(X), t_\epsilon(Y)]). \quad (4.1)$$

If the limit

$$[X, Y]_0 = \lim_{\epsilon \rightarrow 0^+} [X, Y]_\epsilon \quad (4.2)$$

exists for all $X, Y \in U$, then $[_, _]_0$ is a Lie product on U and the Lie algebra $\mathcal{G}_0 = (U, [_, _]_0)$ is called the contraction of \mathcal{G} by t_ϵ and we write $\mathcal{G} \xrightarrow{t(\epsilon)} \mathcal{G}_0$.

There is an analogous definition [12] for the case that the limit (4.2) is meaningful only on a sequence:

Definition 2. Let U be a complex or real vector space, $\mathcal{G} = (U, [_, _])$ a Lie algebra with Lie product $[_, _]$. For any $n \in \mathbb{N}$ let $t_n \in \text{Aut}(U)$ and for every $X, Y \in U$ we define

$$[X, Y]_n = t_n^{-1}([t_n(X), t_n(Y)]). \quad (4.3)$$

If the limit

$$[X, Y]_\infty = \lim_{n \rightarrow \infty} [X, Y]_n \quad (4.4)$$

exists for all $X, Y \in U$, then $[_, _]_\infty$ is a Lie product on U and the Lie algebra $\mathcal{G}_\infty = (U, [_, _]_\infty)$ is called the contraction of \mathcal{G} by t_n and we write $\mathcal{G} \xrightarrow{t_n} \mathcal{G}_\infty$.

Specific examples of contractions of Lie algebras can be found in e.g., [1, 3, 13, 14].

For the $so(4) \rightarrow iso(3)$ case we define the contraction transformation to be $t_\epsilon(M_i) = M_i$, $t_\epsilon(N_i) = \epsilon N_i$ for every $i \in \{1, 2, 3\}$. Then we easily see that:

$$[M_i, M_j]_0 = i\epsilon_{ijk}M_k \quad (4.5)$$

$$[N_i, N_j]_0 = 0 \quad (4.6)$$

$$[M_i, N_j]_0 = i\epsilon_{ijk}N_k \quad (4.7)$$

We recall that

$$[J_i, J_j] = i\epsilon_{ijk}J_k \quad (4.8)$$

$$[P_i, P_j] = 0 \quad (4.9)$$

$$[J_i, P_j] = i\epsilon_{ijk}P_k \quad (4.10)$$

and we see that the linear map ψ , from the contracted Lie algebra, $so(4)_0$ to $iso(3)$ which is defined by $\psi(M_i) = J_i$, $\psi(N_i) = P_i$ for $i \in \{1, 2, 3\}$ is a Lie algebra isomorphism.

4.2. convergence of the matrix elements.

Fix a representation $\eta_{p_1^2, C_1}$ of $iso(3)$ and define

$$j_0^1 = \sqrt{\frac{C_1^2}{p_1^2}} \quad (4.11)$$

We define a sequence of representations which consists of some of the representations $\rho_{(F,G)}$, as follows. We take those $\rho_{(F,G)}$ such that the value of their j_0 parameter equals j_0^1 and such that $sgn(G) = sgn(C_1)$. There is exactly one irreducible representation for each admissible value of n , where the admissible values of n are $I = \{j_0^1, j_0^1 + 1, j_0^1 + 2, \dots\}$. We can describe this sequence by $\{(\rho_{(F(n), G(n))}, V_{F(n), G(n)})\}_{n \in I}$ where

$$G(n) = sgn(C_1)j_0^1(n+1) \quad (4.12)$$

$$F(n) = \frac{(j_0^1)^2 + (n+1)^2 - 1}{2} \quad (4.13)$$

Before we prove the convergence of matrix elements we need the following technical proposition:

Proposition 1. *For*

$$\epsilon_n = \sqrt{\frac{p_1^2}{2F(n)}} = \sqrt{\frac{p_1^2}{j_{0,1}^2 + (n+1)^2 - 1}} \quad (4.14)$$

the following hold

$$\lim_{n \rightarrow \infty} \epsilon_n \beta_j^{F(n), G(n)} = \tilde{\beta}_j^{p_1^2, C_1} \quad (4.15)$$

$$\lim_{n \rightarrow \infty} \epsilon_n \alpha_j^{F(n), G(n)} = \tilde{\alpha}_j^{p_1^2, C_1} \quad (4.16)$$

Proof. For (4.15) we observe that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \epsilon_n \beta_j^{F(n), G(n)} &= \lim_{n \rightarrow \infty} \sqrt{\frac{p_1^2}{2F(n)} \frac{G(n)}{j(j+1)}} = \\
\lim_{n \rightarrow \infty} \sqrt{\frac{p_1^2}{(j_0^1)^2 + (n+1)^2 - 1}} \frac{(\text{sign}(C_1) j_0^1 (n+1))}{j(j+1)} &= \sqrt{p_1^2} \frac{(\text{sign}(C_1) j_0^1)}{j(j+1)} \\
&\stackrel{(4.11)}{=} \sqrt{p_1^2} \frac{(\text{sign}(C_1) \sqrt{\frac{C_1^2}{p_1^2}})}{j(j+1)} = \frac{C_1}{j(j+1)} = \tilde{\beta}_j^{p_1^2, C_1}
\end{aligned} \tag{4.17}$$

(4.16) is obtained similarly. \square

Theorem 1. For any $|_{j, m_j}^{F(n), G(n)}\rangle, |_{\tilde{j}, \tilde{m}_{\tilde{j}}}^{F(n), G(n)}\rangle \in V_{F(n), G(n)}$ and any $X \in so(4)$

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left\langle |_{\tilde{j}, \tilde{m}_{\tilde{j}}}^{F(n), G(n)} | \rho_{F(n), G(n)}(t_n(X)) |_{j, m_j}^{F(n), G(n)} \right\rangle &= \\
\left\langle |_{\tilde{j}, \tilde{m}_{\tilde{j}}}^{p_1^2, C_1} | \eta_{p_1^2, C_1}(\psi(X)) |_{j, m_j}^{p_1^2, C_1} \right\rangle
\end{aligned} \tag{4.18}$$

where $t_n = t(\epsilon_n)$.

Proof. We note that from linearity it is enough to prove that (4.18) holds for $X \in \{M_+, M_-, M_3, N_+, N_-, N_3\}$. We have:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left\langle |_{j, m_j}^{F(n), G(n)} | \rho_{F(n), G(n)}(t_n(M_3)) |_{j, m_j}^{F(n), G(n)} \right\rangle &= \\
\lim_{n \rightarrow \infty} \left\langle |_{j, m_j}^{F(n), G(n)} | m_j |_{j, m_j}^{F(n), G(n)} \right\rangle = m_j &= \left\langle |_{j, m_j}^{p_1^2, C_1} | \eta_{p_1^2, C_1}(\psi(M_3)) |_{j, m_j}^{p_1^2, C_1} \right\rangle
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left\langle |_{j, m_j \pm 1}^{F(j), G(j)} | \rho_{F(n), G(n)}(t_n(M_{\pm})) |_{j, m_j}^{F(n), G(n)} \right\rangle &= \\
\lim_{n \rightarrow \infty} \sqrt{(j \mp m_j)(j \pm m_j + 1)} &= \sqrt{(j \mp m_j)(j \pm m_j + 1)} = \\
= \left\langle |_{j, m_j \pm 1}^{p_1^2, C_1} | \eta_{p_1^2, C_1}(\psi(M_{\pm})) |_{j, m_j}^{p_1^2, C_1} \right\rangle
\end{aligned} \tag{4.20}$$

$$\begin{aligned}
\lim_{n \rightarrow \infty} \sum_{k=-1}^1 \langle {}^{F(n),G(n)}_{j+k,m_j} | \rho_{F(n),G(n)}(t_{\epsilon_n}(N_3)) | {}^{F(n),G(n)}_{j,m_j} \rangle = \quad (4.21) \\
\lim_{n \rightarrow \infty} \epsilon_n \alpha_j^{F(n),G(n)} \sqrt{(j+m_j)(j-m_j)} + \epsilon_n \beta_j^{F(n),G(n)} m_j | {}^{F(n),G(n)}_{j,m_j} \rangle + \\
\epsilon_n \alpha_{j+1}^{F(n),G(n)} \sqrt{(j+m_j+1)(j-m_j+1)} = \\
\tilde{\alpha}_j^{p_1^2, C_1} \sqrt{(j+m_j)(j-m_j)} + \tilde{\beta}_j^{p_1^2, C_1} m_j | {}^{F(n),G(n)}_{j,m_j} \rangle + \\
\tilde{\alpha}_{j+1}^{p_1^2, C_1} \sqrt{(j+m_j+1)(j-m_j+1)} = \sum_{k=-1}^1 \langle {}^{p_1^2, C_1}_{j+k,m_j} | \eta_{p_1^2, C_1}(\psi(N_3)) | {}^{p_1^2, C_1}_{j,m_j} \rangle
\end{aligned}$$

where we have used proposition 1. Similarly:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \sum_{k=-1}^1 \langle {}^{F(n),G(n)}_{j+k,m_j \pm 1} | \rho_{F(n),G(n)}(t_{\epsilon_n}(N_{\pm})) | {}^{F(n),G(n)}_{j,m_j} \rangle = \quad (4.22) \\
\lim_{n \rightarrow \infty} \pm \epsilon_n \alpha_j^{F(n),G(n)} \sqrt{(j \mp m_j)(j \mp m_j - 1)} + \\
\epsilon_n \beta_j^{F(n),G(n)} \sqrt{(j \mp m_j)(j \pm m_j + 1)} \mp \\
\epsilon_n \alpha_{j+1}^{F(n),G(n)} \sqrt{(j \pm m_j + 1)(j \pm m_j + 2)} = \pm \tilde{\alpha}_j^{p_1^2, C_1} \sqrt{(j \mp m_j)(j \mp m_j - 1)} + \\
\tilde{\beta}_j^{p_1^2, C_1} \sqrt{(j \mp m_j)(j \pm m_j + 1)} \mp \tilde{\alpha}_{j+1}^{p_1^2, C_1} \sqrt{(j \pm m_j + 1)(j \pm m_j + 2)} = \\
\sum_{k=-1}^1 \langle {}^{p_1^2, C_1}_{j+k,m_j \pm 1} | \eta_{p_1^2, C_1}(\psi(N_{\pm})) | {}^{p_1^2, C_1}_{j,m_j} \rangle
\end{aligned}$$

All the other matrix elements vanish and obviously satisfy (4.18). \square

4.3. Graphical representation of the contraction process.

In figure 1 each point with coordinates (k, l) represents the irreducible representation of $so(4)$ which we denoted by $\rho^k \otimes \rho^l$. In each "diagonal" line, j_0 is constant and equal to the value of $|k - l|$ (those are the lines $k - l = \pm j_0$ in the k, l plane). Going along each diagonal line in the direction of the arrow (which is equivalent to taking ϵ_n to zero) we are increasing the value of n by one unit at each step, and this is the picture of the contraction. The solid, dashed and dotted diagonal lines correspond to contractions toward $\eta_{p^2, C}$ with their j_0 parameter equal to 0, 1 and $\frac{1}{2}$ respectively.

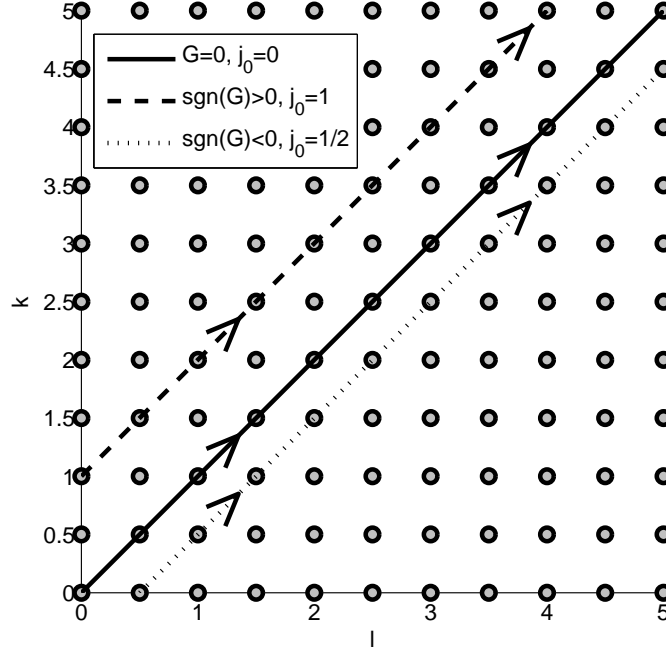


FIGURE 1. Grid of $so(4)$ finite dimensional irreducible representations and some of their contractions. Note the sign conventions indicated in the box (upper left insert).

5. DISCUSSION

The four-dimensional rotation group, $SO(4)$ occurs as a symmetry group of a physical system. The best known example is as the symmetry group of the Hydrogen atom. The group of isometries of the three-dimensional space, \mathbb{R}^3 i.e., the Euclidean group $ISO(3)$ is another group that is naturally related to many physical systems. Among others, $ISO(3)$ is a subgroup of both Poincaré group and Galilei group. The relation between $SO(4)$ and $ISO(3)$ was only partially studied, e.g., [15, 16, 17].

In another work [18, 19] we give a definition for contraction of Lie algebra representations using the notion of direct limit. We also show there that the convergence of matrix elements implies the convergence in norm of the sequence of operators. This shows that the contraction we obtained here is also a contraction according to the definition in [18].

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